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FREQUENCIES OF PIEZOELECTRICALLY FORCED VIBRATIONS OF ELECTRODED, DOUBLY ROTATED, QUARTZ PLATES†

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Abstract—Two-dimensional equations of motion of piezoelectric crystal plates, obtained from the three-dimensional equations of linear piezoelectricity by expansion in power series of the thickness coordinate of the plate, are solved for forced vibrations of electroded SC-cut quartz plates. Results of computations are given for frequencies of simple thickness modes of vibration, for the dispersion of straight-crested waves and for frequencies of vibration of a strip, along with its dimensional ratios for minimal coupling of the fundamental thickness-shear mode with overtones of flexure, face-shear and thickness-twist.

1. INTRODUCTION

A review of vibrational properties of doubly rotated quartz plates may be found in a recent article by Kusters[1]. Solutions of the three-dimensional equations of piezoelectricity for the simple thickness modes of vibration of doubly rotated plates appear in Tiersten's book[2] and in a comprehensive article by Ballato[3]. Simple thickness modes are those in which the oscillatory displacements are functions of only the thickness coordinate of the plate. Solutions of two dimensional plate-equations for doubly rotated quartz strips, in which the displacements depend on a coordinate in the plane of the stip, were presented by Lee and Wu[4] for the purely elastic case. Some of their results are extended, in the present paper, to account for the effects of piezoelectric coupling and the mass of electrode coatings.

In the following Section the three-dimensional, linear equations of piezoelectricity are exhibited. The equations are solved, in Section 3, for the simple thickness modes and the results of computations of the first three frequencies are given for the electroded SC-cut. A brief review of the derivation of two dimensional plate-equations by expansion in power series is given in Section 4 and restricted to the first order in Section 5. The solution of the two-dimensional equations for the simple thickness modes and the solution of the equations governing the correction factors for the simple thickness frequencies are described in Section 6. Section 7 contains the derivation of the dispersion relation for forced, straight-crested waves in the electroded plate and the graphical presentation of the results of computations of the branches for the SC-cut. The final Section includes the solution for the forced vibrations of an electroded strip with mixed edge-conditions and the graphs of portions of the frequency spectrum. Also, in Section 8 is a sketch which transforms a frequency spectrum for mixed edge-conditions approximately to one for free edges. The results exhibit discrete ranges of dimensional ratios favorable to the avoidance of coupling of the fundamental thickness-shear mode with flexure, extension and face-shear overtones and the associated activity dips. An appendix contains formulas for computing the material constants of quartz referred to doubly rotated axes along with the results for the SC-cut.

2. THREE-DIMENSIONAL EQUATIONS OF PIEZOELECTRICITY

We begin with a brief review of the three-dimensional, linear equations of piezoelectricity[2] from which are to be deduced two-dimensional plate-equations and frequencies of simple thickness modes which are required for the computation of correction factors appearing in the two-dimensional equations.

The three-dimensional field equations are the stress equations of motion and the charge equation of electrostatics:

$$T_{ij,i} = \rho \ddot{u}_j, \quad D_{i,i} = 0, \tag{1}$$

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where ρ and the T_{ij}, u_j and D_i are, respectively, the mass density and the components of stress, mechanical displacement and electric displacement.

The constitutive equations are

$$T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i = e_{ijk}S_{jk} + \epsilon_{ij}E_j, \tag{2}$$

where the c_{ijkl}, e_{kij} and ϵ_{ij} are the components of elastic stiffness, piezoelectric strain constant and dielectric permittivity, respectively. S_{ij} and E_i are the components of strain and electric field which are expressed in terms of the u_i and the electric potential ϕ by

$$S_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}), \quad E_i = -\phi_{,i}. \tag{3}$$

Combining (1)–(3), we have the equations of motion,

$$\begin{aligned} c_{ijkl}u_{k,li} + e_{kij}\phi_{,ki} &= \rho\ddot{u}_j, \\ e_{kij}u_{k,ij} - \epsilon_{ij}\phi_{,ij} &= 0, \end{aligned} \tag{4}$$

which can be derived from a variational principle, for a region V bounded by a surface S with outward normal $\mathbf{n}$:

$$\delta\int_{t_0}^{t_1} \mathrm{d}t \int_V (K-H)\,\mathrm{d}V + \int_{t_0}^{t_1} \mathrm{d}t \int_S (t_j\delta u_j + \sigma\delta\phi)\,\mathrm{d}S = 0 \tag{5}$$

where the t_j and σ are the surface traction and surface charge. K and H are the kinetic energy and electric enthalpy densities:

$$K = \frac{1}{2}\rho\dot{u}_i\dot{u}_i, \quad H = \frac{1}{2}c_{ijkl}S_{ij}S_{kl} - \frac{1}{2}\epsilon_{ij}E_iE_j - e_{ijk}E_iS_{jk}, \tag{6}$$

from which

$$T_{ij} = \partial H/\partial S_{ij}, \quad D_i = -\partial H/\partial E_i. \tag{7}$$

Upon substitution of (6) and (7) in (5), the latter can be converted to the variational equation of motion:

$$\int_{t_0}^{t_1} \mathrm{d}t \int_V [(T_{ij,i} - \rho\ddot{u}_j)\delta u_j + D_{i,i}\delta\phi]\,\mathrm{d}V + \int_{t_0}^{t_1} \mathrm{d}t \int_S [(t_j - n_iT_{ij})\delta u_j + (\sigma - n_iD_i)\delta\phi]\,\mathrm{d}S = 0, \tag{8}$$

which produces the field equations (1) and the boundary conditions

$$n_iT_{ij} = t_j \quad \text{or} \quad u_i = \bar{u}_i \quad \text{on} \quad S, \tag{9}$$

where $\bar{u}_i$ is the surface displacement, and

$$n_iD_i = \sigma \quad \text{or} \quad \phi = \bar{\phi} \quad \text{on} \quad S, \tag{10}$$

where $\bar{\phi}$ is the surface potential. As an alternative to (9), a component of n_iT_{ij} and the resultant of u_i in the plane at right angles, or *vice versa*, may be specified.

3. SIMPLE THICKNESS-MODES

Simple thickness-modes of vibration of a plate are those in which the three components of displacement are independent of the coordinates parallel to the middle plane of the plate.

We consider an infinite plate bounded by surfaces at $x_2 = \pm b$ which are coated with electrodes each of thickness $2b'$ and mass density ρ'. A uniform alternating voltage $V\,e^{i\omega t}$ is applied to the electrodes so that the voltage drop across the thickness of the plate is $2V$. The response of the plate is independent of x_1 and x_3 whence (4), (in the reduced notation whereby pairs of indices 11, 22, 33, 23 or 32, 31 or 13, 12 or 21 become 1, 2, 3, 4, 5, 6, respectively) reduce to

$$\begin{aligned}
c_{66}u_{1,22} + c_{26}u_{2,22} + c_{46}u_{3,22} + e_{26}\phi_{,22} &= \rho\ddot{u}_1,\\
c_{26}u_{1,22} + c_{22}u_{2,22} + c_{24}u_{3,22} + e_{22}\phi_{,22} &= \rho\ddot{u}_2,\\
c_{46}u_{1,22} + c_{24}u_{2,22} + c_{44}u_{3,22} + e_{24}\phi_{,22} &= \rho\ddot{u}_3,\\
e_{26}u_{1,22} + e_{22}u_{2,22} + e_{24}u_{3,22} - \epsilon_{22}\phi_{,22} &= 0.
\end{aligned} \tag{11}$$

The boundary conditions (9) and (10), on $x_2 = \pm b$, become

$$(T_6, T_2, T_4) = \mp 2\rho' b'(\ddot{u}_1, \ddot{u}_2, \ddot{u}_3), \quad \phi = \pm V\,e^{i\omega t} \tag{12}$$

or

$$\begin{aligned}
c_{66}u_{1,2} + c_{26}u_{2,2} + c_{46}u_{3,2} + e_{26}\phi_{,2} &= \mp 2\rho' b'\ddot{u}_1,\\
c_{26}u_{1,2} + c_{22}u_{2,2} + c_{24}u_{3,2} + e_{22}\phi_{,2} &= \mp 2\rho' b'\ddot{u}_2,\\
c_{46}u_{1,2} + c_{24}u_{2,2} + c_{44}u_{3,2} + e_{24}\phi_{,2} &= \mp 2\rho' b'\ddot{u}_3,\\
\phi &= \pm V\,e^{i\omega t}.
\end{aligned} \tag{13}$$

From the fourth of (11),

$$\phi = (e_{26}u_1 + e_{22}u_2 + e_{24}u_3)/\epsilon_{22} + Ax_2 + B; \tag{14}$$

but the constant B may be omitted as a constant ϕ produces no electric field.

Now, substitute (14) in the first three of (11) and get the same form of equations except with c_{pq} replaced by $\bar{c}_{pq}$, where

$$\bar{c}_{pq} = c_{pq} + e_{2p}e_{2q}/\epsilon_{22}. \tag{15}$$

Thus:

$$\begin{aligned}
\bar{c}_{66}u_{1,22} + \bar{c}_{26}u_{2,22} + \bar{c}_{46}u_{3,22} &= \rho\ddot{u}_1,\\
\bar{c}_{26}u_{1,22} + \bar{c}_{22}u_{2,22} + \bar{c}_{24}u_{3,22} &= \rho\ddot{u}_2,\\
\bar{c}_{46}u_{1,22} + \bar{c}_{24}u_{2,22} + \bar{c}_{44}u_{3,22} &= \rho\ddot{u}_3.
\end{aligned} \tag{16}$$

Now, take

$$u_i = A_i \sin \eta x_2\, e^{i\omega t}, \quad \phi = \phi_0 \sin \eta x_2\, e^{i\omega t} \tag{17}$$

and substitute in (16) to get

$$\begin{aligned}
(\bar{\eta}^2 - \bar{\Omega}^2)A_1 + \tilde{c}_{26}\bar{\eta}^2 A_2 + \tilde{c}_{46}\bar{\eta}^2 A_3 &= 0,\\
\tilde{c}_{26}\bar{\eta}^2 A_1 + (\tilde{c}_{22}\bar{\eta}^2 - \bar{\Omega}^2)A_2 + \tilde{c}_{24}\bar{\eta}^2 A_3 &= 0\\
\tilde{c}_{46}\bar{\eta}^2 A_1 + \tilde{c}_{24}\bar{\eta}^2 A_2 + (\tilde{c}_{44}\bar{\eta}^2 - \bar{\Omega}^2)A_3 &= 0
\end{aligned} \tag{18}$$

where

$$\bar{\eta} = \eta b, \quad \bar{\Omega}^2 = \rho\omega^2 b^2/\bar{c}_{66}, \quad \tilde{c}_{pq} = \bar{c}_{pq}/c_{66}. \tag{19}$$

The determinant of the coefficients of the A_i in (18), set equal to zero, is a bicubic equation in η with three roots η_j^2, $j = 1, 2, 3$. Then (17) can be extended to

$$u_i = \sum_{j=1}^{3} A_{ij} \sin \eta_j x_2 \tag{20}$$

and (14) to

$$\phi = \sum_{j=1}^{3} \epsilon_{22}^{-1}(e_{26}A_{1j} + e_{22}A_{2j} + e_{24}A_{3j}) \sin \eta_j x_2 + Ax_2, \tag{21}$$

with $e^{i\omega t}$ omitted here and in the sequel.

For each η_j, define

$$\alpha_{2j} = A_{2j}/A_{1j}, \quad \alpha_{4j} = A_{3j}/A_{1j}, \quad \alpha_{6j} = 1, \quad B_j = A_{1j}/b. \tag{22}$$

Then

$$\begin{aligned} u_1 &= \sum_j B_j b \alpha_{6j} \sin \eta_j x_2, \\ u_2 &= \sum_j B_j b \alpha_{2j} \sin \eta_j x_2, \\ u_3 &= \sum_j B_j b \alpha_{4j} \sin \eta_j x_2, \\ \phi &= \sum_j B_j b \epsilon_{22}^{-1}(e_{26}\alpha_{6j} + e_{22}\alpha_{2j} + e_{24}\alpha_{4j}) \sin \eta_j x_2 + Ax_2. \end{aligned} \tag{23}$$

We may find α_{2j} and α_{4j} from the second and third of (18) with η replaced by η_j:

$$\begin{aligned} (\bar{c}_{22}\bar{\eta}_j^2 - \bar{\Omega}^2)\alpha_{2j} + \bar{c}_{24}\bar{\eta}_j^2\alpha_{4j} &= -\bar{c}_{26}\bar{\eta}_j^2, \\ \bar{c}_{24}\bar{\eta}_j^2\alpha_{2j} + (\bar{c}_{44}\bar{\eta}_j^2 - \bar{\Omega}^2)\alpha_{4j} &= -\bar{c}_{46}\bar{\eta}_j^2, \end{aligned} \tag{24}$$

from which

$$\begin{aligned} \alpha_{2j} &= [(\bar{c}_{24}\bar{c}_{46} - \bar{c}_{26}\bar{c}_{44})\bar{\eta}_j^4 + \bar{c}_{26}\bar{\eta}_j^2\bar{\Omega}^2]/\alpha, \\ \alpha_{4j} &= [(\bar{c}_{24}\bar{c}_{26} - \bar{c}_{46}\bar{c}_{22})\bar{\eta}_j^4 + \bar{c}_{46}\bar{\eta}_j^2\bar{\Omega}^2]/\alpha, \\ \alpha &= (\bar{c}_{22}\bar{\eta}_j^2 - \bar{\Omega}^2)(\bar{c}_{44}\bar{\eta}_j^2 - \bar{\Omega}^2) - \bar{c}_{24}^2\bar{\eta}_j^4. \end{aligned} \tag{25}$$

Upon substituting (25) in (23) and the result in the boundary conditions (13), we find

$$\begin{aligned} \beta_{61}B_1 + \beta_{62}B_2 + \beta_{63}B_3 &= -e_{26}V/b\bar{c}_{66}, \\ \beta_{21}B_1 + \beta_{22}B_2 + \beta_{23}B_3 &= -e_{22}V/b\bar{c}_{66}, \\ \beta_{41}B_1 + \beta_{42}B_2 + \beta_{43}B_3 &= -e_{24}V/b\bar{c}_{66}, \end{aligned} \tag{26}$$

where

$$\beta_{ai} = \sum_b \alpha_{bi}[\bar{c}_{ab}\bar{\eta}_i \cos \bar{\eta}_i - (\delta_{ab}R\bar{\Omega}^2 + \bar{e}_{ab}) \sin \bar{\eta}_i] \tag{27}$$

in which a and b range over 2, 4, 6 and i over 1, 2, 3; δ_{ab} is the Krönecker delta and

$$\bar{e}_{ab} = e_{2a}e_{2b}/\epsilon_{22}\bar{c}_{66}, \quad R = 2\rho' b'/\rho b, \tag{28}$$

i.e. R is the ratio of the mass of both electrodes to the mass of the crystal.

The determinant of the coefficients of the B_i in (26), set equal to zero:

$$|\beta_{ai}| = 0, \tag{29}$$

is a transcendental equation whose roots $\bar{\Omega}$ give the frequencies of the simple thickness-modes. Except for notation, division by $\bar{c}_{pq}$ to make each element dimensionless and the additional term $\delta_{ab}R\bar{\Omega}^2$, which accounts for the inertia of the electrodes, (29) is the same as Tiersten's equation (9.69), ([2], p.92).

For the SC-cut (see the Appendix) and $R = 0.01$, the first three roots of (29), converted from the $\bar{\Omega}$ of (19) to

$$\Omega = (2\omega b/\pi)(\rho/c_{66})^{1/2} = (2\bar{\Omega}/\pi)(\bar{c}_{66}/c_{66})^{1/2}, \tag{30}$$

are

$$\begin{aligned} \Omega_2 &= 1.7512 \\ \Omega_4 &= 1.0235 \\ \Omega_6 &= 0.9304 \end{aligned} \tag{31}$$

where Ω_2, Ω_4, Ω_6 give the fundamental frequencies of the essentially thickness-stretch, -twist, -shear modes, respectively; "essentially" because each of the three modes has contributions from all three components of displacement, u_1, u_2, u_3, one of which predominates in each mode.

4. EXPANSION IN POWER SERIES

The two-dimensional equations of motion of piezoelectric crystal plates, to be used in the sequel, are deduced from the three-dimensional equations by a procedure based on expansions in series of powers of the thickness coordinate of the plate. The process was developed in stages: beginning in 1952[5], extended in 1962[6] and revised in 1972[7]. As revised, the start is with expansions of mechanical displacement and electric potential in series of powers of the thickness coordinate x_2:

$$u_i = \sum_n x_2{}^n u_i^{(n)}, \quad \phi = \sum_n x_2{}^n \phi^{(n)}, \tag{32}$$

where $u_i^{(n)}$ and $\phi^{(n)}$ are independent of x_2. The three-dimensional strain and electric field are, from (3) and (32),

$$S_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}) = \sum_n x_2{}^n S_{ij}^{(n)}, \quad E_i = -\phi_{,i} = \sum_n x_2{}^n E_i^{(n)}, \tag{33}$$

in which the two-dimensional strain and electric field of order n are

$$\begin{aligned} S_{ij}^{(n)} &= \frac{1}{2}[u_{j,i}^{(n)} + u_{i,j}^{(n)} + (n+1)(\delta_{i2}u_j^{(n+1)} + \delta_{2j}u_i^{(n+1)})], \\ E_i^{(n)} &= -\phi_{,i}^{(n)} - (n+1)\delta_{i2}\phi^{(n+1)}. \end{aligned} \tag{34}$$

The kinetic energy density and electric enthalpy density of the plate are defined as

$$\bar{K} = \int_{-b}^{b} K \, dx_2, \quad \bar{H} = \int_{-b}^{b} H \, dx_2, \tag{35}$$

whence, from (6), (32) and (34),

$$\bar{K} = \frac{1}{2}\rho \sum_m \sum_n B_{mn} \dot{u}_j^{(m)} \dot{u}_j^{(n)}$$

$$\bar{H} = \frac{1}{2}\sum_m \sum_n B_{mn}(c_{ijkl} S_{ij}^{(m)} S_{kl}^{(n)} - \epsilon_{ij} E_i^{(m)} E_j^{(n)} - 2e_{ijk} E_i^{(m)} S_{jk}^{(n)}) \tag{36}$$

where

$$B_{mn} = 2b^{m+n+1}/(m+n+1),\ m+n \quad \text{even;} \quad 0,\ m+n \quad \text{odd.} \tag{37}$$

The constitutive equations of order n are

$$T_{ij}^{(n)} = \int_{-b}^{b} x_2{}^n T_{ij}\, \mathrm{d}x_2 = \sum_m B_{mn}(c_{ijkl} S_{kl}^{(m)} - e_{kij} E_k^{(m)}) = \partial \bar{H}/\partial S_{ij}^{(n)},$$

$$D_i^{(n)} = \int_{-b}^{b} x_2{}^n D_i\, \mathrm{d}x_2 = \sum_m B_{mn}(e_{ijk} S_{jk}^{(m)} + \epsilon_{ij} E_j^{(m)}) = -\,\partial \bar{H}/\partial E_i^{(n)}. \tag{38}$$

The variational equation of motion becomes

$$\sum_n \int_{t_0}^{t_1} \mathrm{d}t \int_A (T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + B_n T_j^{(n)} - \rho \sum_m B_{mn} \ddot{u}_j^{(m)}) \delta u_j^{(n)}\, \mathrm{d}A$$
$$+ \sum_n \int_{t_0}^{t_1} \mathrm{d}t \int_A (D_{i,i}^{(n)} - nD_2^{(n-1)} + B_n D^{(n)}) \delta \phi^{(n)}\, \mathrm{d}A$$
$$+ \sum_n \int_{t_0}^{t_1} \mathrm{d}t \oint_C [(B_n t_j^{(n)} - n_c T_{cj}^{(n)}) \delta u_j^{(n)} + (B_n d^{(n)} - n_c D_c^{(n)}) \delta \phi^{(n)}]\, \mathrm{d}s = 0, \tag{39}$$

where the index c ranges over 1 and 3 only, s is the coordinate along the edge curve C,

$$B_n = 2b^{2n+1}/(2n+1) \tag{40}$$

and the face-tractions $T_j^{(n)}$, face charges $D^{(n)}$, edge tractions $t^{(n)}$ and edge charges $d^{(n)}$ are defined by

$$T_j^{(n)} = B_n^{-1}[x_2{}^n T_{2j}]_{-b}^{b}, \quad D^{(n)} = B_n^{-1}[x_2{}^n D_2]_{-b}^{b}$$
$$t_j^{(n)} = B_n^{-1} \int_{-b}^{b} x_2{}^n (n_c T_{cj})_C\, \mathrm{d}x_2, \quad d^{(n)} = B_n^{-1} \int_{-b}^{b} x_2{}^n (n_c D_c)_C\, \mathrm{d}x_2. \tag{41}$$

Finally, the field equations of order n are

$$T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + B_n T_j^{(n)} = \rho \sum_m B_{mn} \ddot{u}_j^{(m)},$$
$$D_{i,i}^{(n)} - nD_2^{(n-1)} + B_n D^{(n)} = 0, \tag{42}$$

with edge conditions, on C,

$$n_c T_{cj}^{(n)} = B_n t_j^{(n)} \quad \text{or} \quad u_j^{(n)} = \bar{u}_j^{(n)} \tag{43}$$

where $\bar{u}_j^{(n)}$ is the edge displacement of order n, and

$$n_c D_c^{(n)} = B_n d^{(n)} \quad \text{or} \quad \phi^{(n)} = \bar{\phi}^{(n)} \tag{44}$$

where $\bar{\phi}^{(n)}$ is the edge potential of order n. As an alternative to (43), a component of $n_i T_{in}^{(n)}$ and the resultant of $u_j^{(n)}$ in the plane at right angles, or *vice versa*, may be specified.

5. FIRST ORDER APPROXIMATION

The equations of the preceding section are now to be restricted so as to exclude simple thickness modes higher than the fundamental stretch, twist and shear modes—which involve only terms of orders $n = 0, 1$. The components of strain and electric field, of these orders, are, from (34),

$$\begin{aligned} S_{11}^{(0)} &= S_1^{(0)} = u_{1,1}^{(0)} & 2S_{23}^{(0)} &= S_4^{(0)} = u_{2,3}^{(0)} + u_3^{(1)} & E_1^{(0)} &= -\phi_{,1}^{(0)} \\ S_{22}^{(0)} &= S_2^{(0)} = u_2^{(1)} & 2S_{31}^{(0)} &= S_5^{(0)} = u_{1,3}^{(0)} + u_{3,1}^{(0)} & E_2^{(0)} &= -\phi^{(1)} \\ S_{33}^{(0)} &= S_3^{(0)} = u_{3,3}^{(0)} & 2S_{12}^{(0)} &= S_6^{(0)} = u_{2,1}^{(0)} + u_1^{(1)} & E_3^{(0)} &= -\phi_{,3}^{(0)} \end{aligned} \tag{45}$$

$$\begin{aligned} S_{11}^{(1)} &= S_1^{(1)} = u_{1,1}^{(1)} & 2S_{23}^{(1)} &= S_4^{(1)} = u_{2,3}^{(1)} + 2u_3^{(2)} & E_1^{(1)} &= -\phi_{,1}^{(1)} \\ S_{22}^{(1)} &= S_2^{(1)} = 2u_2^{(2)} & 2S_{31}^{(1)} &= S_5^{(1)} = u_{1,3}^{(1)} + u_{3,1}^{(1)} & E_2^{(1)} &= -2\phi^{(2)} \\ S_{33}^{(1)} &= S_3^{(1)} = u_{3,3}^{(1)} & 2S_{12}^{(1)} &= S_6^{(1)} = u_{2,1}^{(1)} + 2u_1^{(2)} & E_3^{(1)} &= -\phi_{,3}^{(1)}. \end{aligned} \tag{46}$$

The strain $S_{22}^{(1)}$ is associated with the second thickness-stretch mode—which is not to be included. We set $T_{22}^{(1)}$ and $\ddot{u}_2^{(2)}$ equal to zero so as to permit free development of $S_{22}^{(1)}$ without contribution to the kinetic or potential energy. From (38),

$$T_{ij}^{(1)} = (2b^3/3)(c_{ijkl}S_{kl}^{(1)} - e_{kij}E_k^{(1)}) \tag{47}$$

so that we set

$$T_{22}^{(1)} = (2b^3/3)(c_{22kl}S_{kl}^{(1)} - e_{k22}E_k^{(1)}) = 0. \tag{48}$$

Write these two equations as

$$3T_{ij}^{(1)}/2b^3 = (c_{ijkl}S_{kl}^{(1)} - c_{ij22}S_{22}^{(1)}) + c_{ij22}S_{22}^{(1)} - e_{kij}E_k^{(1)} \tag{49}$$

and

$$S_{22}^{(1)} = -c_{22kl}S_{kl}^{(1)}/c_{2222} + S_{22}^{(1)} + e_{k22}E_k^{(1)}/c_{2222}, \tag{50}$$

respectively. Then substitute the expression for $S_{22}^{(1)}$, in (50), for the $S_{22}^{(1)}$ outside the parentheses in (49) and collect terms to obtain, in place of (47),

$$T_{ij}^{(1)} = (2b^3/3)(c_{ijkl}^{(1)}S_{kl}^{(1)} - e_{kij}^{(1)}E_k^{(1)}) \tag{51}$$

where

$$c_{ijkl}^{(1)} = c_{ijkl} - c_{ij22}c_{22kl}/c_{2222}, \quad e_{kij}^{(1)} = e_{kij} - e_{k22}c_{ij22}/c_{2222}. \tag{52}$$

It may be verified that (51) satisfies (48).

The remaining terms in (46) of order $n = 2$ are $u_1^{(2)}$, $u_3^{(2)}$ and $\phi^{(2)}$. These are to be omitted and $e_{ijk}^{(1)}$ substituted for e_{ijk} in $D_i^{(1)}$.

At this stage, the constitutive equations of the first order approximation are

$$T_{ij}^{(0)} = 2b(c_{ijkl}S_{kl}^{(0)} - e_{kij}E_k^{(0)}), \quad D_i^{(0)} = 2b(e_{ijk}S_{jk}^{(0)} + \epsilon_{ij}E_j^{(0)}), \tag{53}$$

$$T_{ij}^{(1)} = (2b^3/3)(c_{ijkl}^{(1)}S_{kl}^{(1)} - e_{kij}^{(1)}E_k^{(1)}), \quad D_i^{(1)} = (2b^3/3)(e_{ijk}^{(1)}S_{jk}^{(1)} + \epsilon_{ij}E_j^{(1)}),$$

where

$$S_{ij}^{(0)} = \frac{1}{2}(u_{j,i}^{(0)} + u_{i,j}^{(0)} + \delta_{i2}u_j^{(1)} + \delta_{2j}u_i^{(1)}), \quad E_i^{(0)} = -\phi_{,i}^{(0)} - \delta_{i2}\phi^{(1)}$$

$$S_{ij}^{(1)} = \frac{1}{2}(u_{j,i}^{(1)} + u_{i,j}^{(1)}), \quad E_i^{(1)} = -\phi_{,i}^{(1)}. \tag{54}$$

In the reduced notation (53) are

$$T_p^{(0)} = 2b(c_{pq}S_q^{(0)} - e_{kp}E_k^{(0)}), \quad D_i^{(0)} = 2b(e_{ip}S_p^{(0)} + \epsilon_{ij}E_j^{(0)}),$$

$$T_p^{(1)} = (2b^3/3)(c_{pq}^{(1)}S_q^{(1)} - e_{kp}^{(1)}E_k^{(1)}), \quad D_i^{(1)} = (2b^3/3)(e_{ip}^{(1)}S_p^{(1)} + \epsilon_{ij}E_j^{(1)}) \tag{55}$$

and the corresponding electric enthalpy density is

$$\bar{H} = b(c_{pq}S_p^{(0)}S_q^{(0)} - \epsilon_{ij}E_i^{(0)}E_j^{(0)} - 2e_{kp}E_k^{(0)}S_p^{(0)})$$
$$+(b^3/3)(c_{pq}^{(1)}S_p^{(1)}S_q^{(1)} - \epsilon_{ij}E_i^{(1)}E_j^{(1)} - 2\,e_{kp}^{(1)}E_k^{(1)}S_p^{(1)}). \tag{56}$$

It is important that the simple thickness frequencies from the approximate equations match exactly those from the three-dimensional equations. As the approximate equations now stand, the match is not exact owing to the difference between the exact trigonometric distributions of displacements, from (23), and the approximate linear distributions from the early terms of (32). To compensate, correction coefficients are inserted as multipliers of the thickness-stretch and -shear strains $S_2^{(0)}$, $S_4^{(0)}$, $S_6^{(0)}$. Thus

$$S_p^{(0)} \rightarrow \kappa_p S_p^{(0)} \quad \text{(not summed)} \tag{57}$$

where

$$\kappa_p = \begin{cases} \kappa_p, & p = 2, 4, 6 \\ 1, & p = 1, 3, 5 \end{cases} \tag{58}$$

and the three κ_p are to be determined so that the three simple thickness frequencies from the approximate equations match those from the three-dimensional equations.

The revised electric enthalpy density is

$$\bar{H} = b(c_{pq}^{(0)}S_p^{(0)}S_q^{(0)} - \epsilon_{ij}E_i^{(0)}E_j^{(0)} - 2e_{kp}^{(0)}E_k^{(0)}S_p^{(0)})$$
$$+ (b^3/3)(c_{pq}^{(1)}S_p^{(1)}S_q^{(1)} - \epsilon_{ij}E_i^{(1)}E_j^{(1)} - 2e_{kp}^{(1)}E_k^{(1)}S_p^{(1)}), \tag{59}$$

where

$$c_{pq}^{(0)} = \kappa_p\kappa_q c_{pq}, \quad e_{kp}^{(0)} = \kappa_p e_{kp} \quad \text{(not summed)} \tag{60}$$

and the $S_p^{(n)}$ and $E_i^{(n)}$ are

$$\begin{array}{lll} S_1^{(0)} = u_{1,1}^{(0)} & S_4^{(0)} = u_{2,3}^{(0)} + u_3^{(1)} & E_1^{(0)} = -\phi_{,1}^{(0)} \\ S_2^{(0)} = u_2^{(1)} & S_5^{(0)} = u_{1,3}^{(0)} + u_{3,1}^{(0)} & E_2^{(0)} = -\phi^{(1)} \\ S_3^{(0)} = u_{3,3}^{(0)} & S_6^{(0)} = u_{2,1}^{(0)} + u_1^{(1)} & E_3^{(0)} = -\phi_{,3}^{(0)} \end{array} \tag{61}$$

$$\begin{array}{lll} S_1^{(1)} = u_{1,1}^{(1)} & S_4^{(1)} = u_{2,3}^{(1)} & E_1^{(1)} = -\phi_{,1}^{(1)} \\ & S_5^{(1)} = u_{1,3}^{(1)} + u_{3,1}^{(1)} & \\ S_3^{(1)} = u_{3,3}^{(1)} & S_6^{(1)} = u_{2,1}^{(1)} & E_3^{(1)} = -\phi_{,3}^{(1)}. \end{array}$$

The constitutive equations are

$$T_p^{(0)} = \partial\bar{H}/\partial S_p^{(0)} = 2b(c_{pq}^{(0)}S_q^{(0)} - e_{kp}^{(0)}E_k^{(0)}),$$
$$T_p^{(1)} = \partial\bar{H}/\partial S_p^{(0)} = (2b/3)(c_{pq}^{(1)}S_q^{(1)} - e_{kp}^{(1)}E_k^{(1)}),$$
$$D_i^{(0)} = -\partial\bar{H}/\partial E_i^{(0)} = 2b(e_{ip}^{(0)}S_p^{(0)} + \epsilon_{ij}E_j^{(0)}), \tag{62}$$
$$D_i^{(1)} = -\partial\bar{H}/\partial E_i^{(1)} = (2b^3/3)(e_{ip}^{(1)}S_p^{(1)} + \epsilon_{ij}E_j^{(1)});$$

and, finally, the field equations are

$$\begin{aligned} T^{(0)}_{ij,i} + 2bT^{(0)}_j &= 2b\rho\ddot{u}^{(0)}_j, \\ T^{(1)}_{ij,i} - T^{(0)}_{2j} + (2b^3/3)T^{(1)}_j &= (2b^3/3)\rho\ddot{u}^{(1)}_j, \\ D^{(0)}_{i,i} + 2bD^{(0)} &= 0, \\ D^{(1)}_{i,i} - D^{(0)}_2 + (2b^3/3)D^{(1)} &= 0. \end{aligned} \tag{63}$$

The inclusion of the thickness-stretch mode, in (59)–(63), is what distinguishes them from the equations in [7].

6. CORRECTION FACTORS

In this section the procedure is established for computing the values of the correction factors κ_a, $a = 2, 4, 6$, so as to make the simple thickness-frequencies, from the first order equations, the same as the corresponding ones obtained in Section 3 from the three-dimensional equations.

To find the simple thickness-frequencies from the first-order equations, we set

$$u^{(0)}_j = 0, \quad u^{(1)}_j = A^0_j\, e^{i\omega t}, \quad \phi^{(0)} = 0, \quad \phi^{(1)} = b^{-1}V\, e^{i\omega t} \tag{64}$$

in the equations of motion.

In general, from (41),

$$T^{(n)}_j = B^{-1}_n[x_2{}^n T_{2j}]^b_{-b}$$

and, for electrode coatings of density ρ' and half-thickness b',

$$T_{2j}]_{\pm b} = \mp 2\rho' b' \ddot{u}_j]_{\pm b}. \tag{65}$$

Then

$$\begin{aligned} T^{(0)}_j &= (2b)^{-1}[\mp 2\rho' b' \ddot{u}_j]_{\pm b} = -R\rho\ddot{u}^{(0)}_j, \\ T^{(1)}_j &= (3/2b^3)[\mp x_2 2\rho' b' \ddot{u}_j]_{\pm b} = -3R\rho\ddot{u}^{(1)}_j \end{aligned} \tag{66}$$

where, again, $R = 2\rho' b'/\rho b$.

With (64) and (66), the stress equations of motion (63) reduce to

$$\begin{aligned} (T^{(0)}_6, T^{(0)}_2, T^{(0)}_4) &= -(2b^3/3)(1+3R)\rho(\ddot{u}^{(1)}_1, \ddot{u}^{(1)}_2, \ddot{u}^{(1)}_3) \\ D^{(0)}_2 &= (2b^3/3)D^{(1)}, \end{aligned} \tag{67}$$

in which

$$\begin{aligned} T^{(0)}_6 &= 2b(c^{(0)}_{66}u^{(1)}_1 + c^{(0)}_{26}u^{(1)}_2 + c^{(0)}_{46}u^{(1)}_3 + e^{(0)}_{26}\phi^{(1)}), \\ T^{(0)}_2 &= 2b(c^{(0)}_{26}u^{(1)}_1 + c^{(0)}_{22}u^{(1)}_2 + c^{(0)}_{24}u^{(1)}_3 + e^{(0)}_{22}\phi^{(1)}), \\ T^{(0)}_4 &= 2b(c^{(0)}_{46}u^{(1)}_1 + c^{(0)}_{24}u^{(1)}_2 + c^{(0)}_{44}u^{(1)}_3 + e^{(0)}_{24}\phi^{(1)}), \\ D^{(0)}_2 &= 2b(e^{(0)}_{26}u^{(1)}_1 + e^{(0)}_{22}u^{(1)}_2 + e^{(0)}_{24}u^{(1)}_3 - \epsilon_{22}\phi^{(1)}). \end{aligned} \tag{68}$$

The last of (67) and (68) contribute to a formula for the current, per unit area, through the crystal:

$$I = (2b^3/3)\, dD^{(1)}/dt. \tag{69}$$

From (64) and the first three of (67) and (68), we find

$$\begin{aligned}(1-\bar{\Omega}^2/\bar{\kappa}_6^{\,2})\kappa_6A_1^{\,0}+\bar{c}_{26}\kappa_2A_2^{\,0}+\bar{c}_{46}\kappa_4A_3^{\,0}&=-e_{26}V/bc_{66},\\ \bar{c}_{26}\kappa_6A_1^{\,0}+(\bar{c}_{22}-\bar{\Omega}^2/\bar{\kappa}_2^{\,2})\kappa_2A_2^{\,0}+\bar{c}_{24}\kappa_4A_3^{\,0}&=-e_{22}V/bc_{66},\\ \bar{c}_{46}\kappa_6A_1^{\,0}+\bar{c}_{24}\kappa_2A_2^{\,0}+(\bar{c}_{44}-\bar{\Omega}^2/\bar{\kappa}_4^{\,2})\kappa_4A_3^{\,0}&=-e_{24}V/bc_{66},\end{aligned}\tag{70}$$

where

$$\bar{c}_{pq}=c_{pq}/c_{66},\quad \bar{\kappa}_a^{\,2}=12\bar{\kappa}_a^{\,2}/\pi^2,\quad \bar{\Omega}^2=(1+3R)\omega^2/(\pi^2c_{66}/4\rho b^2).\tag{71}$$

The determinant of the coefficients of the A_j^0, in (70), set equal to zero, yields

$$A\bar{\Omega}^6+B\bar{\Omega}^4+C\bar{\Omega}^2+D=0,\tag{72}$$

where

$$\begin{aligned}A&=1/\bar{\kappa}_2^{\,2}\bar{\kappa}_4^{\,2}\bar{\kappa}_6^{\,2},\\ B&=-(\bar{c}_{22}\bar{\kappa}_2^{\,2}+\bar{c}_{44}\bar{\kappa}_4^{\,2}+\bar{\kappa}_6^{\,2})/\bar{\kappa}_2^{\,2}\bar{\kappa}_4^{\,2}\bar{\kappa}_6^{\,2},\\ C&=(\bar{c}_{44}-\bar{c}_{46}^2)/\bar{\kappa}_2^{\,2}+(\bar{c}_{22}-\bar{c}_{26}^2)/\bar{\kappa}_4^{\,2}+(\bar{c}_{22}\bar{c}_{44}-c_{24}^2)/\bar{\kappa}_6^{\,2},\\ D&=-(\bar{c}_{22}\bar{c}_{44}+2\bar{c}_{26}\bar{c}_{24}\bar{c}_{46}-\bar{c}_{22}c_{46}^2-\bar{c}_{44}\bar{c}_{26}^2-\bar{c}_{24}^2).\end{aligned}\tag{73}$$

The bicubic (72) must be satisfied by each of

$$\bar{\Omega}_a^{\,2}=\Omega_a^{\,2}(1+3R),\quad a=2,4,6\tag{74}$$

where the Ω_a are the exact roots from (29). Thus, we have the three simultaneous, nonlinear equations on the κ_a:

$$\begin{aligned}A\bar{\Omega}_2^{\,6}+B\bar{\Omega}_2^{\,4}+C\bar{\Omega}_2^{\,2}+D&=0,\\ A\bar{\Omega}_4^{\,6}+B\bar{\Omega}_4^{\,4}+C\bar{\Omega}_4^{\,2}+D&=0,\\ A\bar{\Omega}_6^{\,6}+B\bar{\Omega}_6^{\,4}+C\bar{\Omega}_6^{\,2}+D&=0.\end{aligned}\tag{75}$$

We find, from (75),

$$\begin{aligned}A&=-D/\bar{\Omega}_2^{\,2}\bar{\Omega}_4^{\,2}\bar{\Omega}_6^{\,2},\\ B&=-A(\bar{\Omega}_2^{\,2}+\bar{\Omega}_4^{\,2}+\bar{\Omega}_6^{\,2}),\\ C&=A(\bar{\Omega}_2^{\,2}\bar{\Omega}_4^{\,2}+\bar{\Omega}_4^{\,2}\bar{\Omega}_6^{\,2}+\bar{\Omega}_6^{\,2}\bar{\Omega}_2^{\,2})\end{aligned}\tag{76}$$

and note that the right hand sides of (76) are independent of the κ_a. From the third of (73) and of (76), $\bar{\kappa}_2$ may be expressed in terms of $\bar{\kappa}_4$ and $\bar{\kappa}_6$ (and the Ω_a and $\bar{c}_{pq}$); and this expression may be used to eliminate $\bar{\kappa}_2$ from the second of (76), leaving that equation as a quadratic in $1/\bar{\kappa}_4^{\,2}$ with coefficients functions of $\bar{\kappa}_6$. Then $1/\bar{\kappa}_4^{\,2}$ can be obtained as the algebraically larger root of the quadratic—a function of $\bar{\kappa}_6$, say

$$1/\bar{\kappa}_4^{\,2}=f(\bar{\kappa}_6).\tag{77}$$

With this and the third of (76), we may express $\bar{\kappa}_2$ in terms of $\bar{\kappa}_6$ alone, say

$$1/\bar{\kappa}_2^{\,2}=g(\bar{\kappa}_6).\tag{78}$$

Upon substituting (77) and (78) in the first of (76), we have

$$f(\bar{\kappa}_6)g(\bar{\kappa}_6)/\bar{\kappa}_6^{\,2}+D/\bar{\Omega}_2^{\,2}\bar{\Omega}_4^{\,2}\bar{\Omega}_6^{\,2}=0,\tag{79}$$

an equation on $\bar{\kappa}_6$, alone, which may be solved, iteratively, for $\bar{\kappa}_6$; following which $\bar{\kappa}_2$ and $\bar{\kappa}_4$ may be obtained from (77) and (78). For the SC-cut and $R = 0.01$, we find

$$\begin{aligned} \bar{\kappa}_2 &= 1.004933, \\ \bar{\kappa}_4 &= 1.005039, \\ \bar{\kappa}_6 &= 1.005083. \end{aligned} \tag{80}$$

7. DISPERSION RELATION

We consider, again, an electroded, infinite plate under a uniform voltage drop $V\,e^{i\omega t}/b$ across the thickness; but, in addition to the forced, simple thickness-modes, of the preceding section, we now permit variation of displacements along x_1:

$$\begin{aligned} &u_1^{(0)} = A_1 b \sin \xi x_1\, e^{i\omega t}, \quad u_1^{(1)} = A_4 \cos \xi x_1\, e^{i\omega t}, \\ &u_2^{(0)} = A_2 b \sin \xi x_1\, e^{i\omega t}, \quad u_2^{(1)} = A_5 \cos \xi x_1\, e^{i\omega t}, \\ &u_3^{(0)} = A_3 b \sin \xi x_1\, e^{i\omega t}, \quad u_3^{(1)} = A_6 \cos \xi x_1\, e^{i\omega t}, \\ &\phi^{(0)} = A_7 b \sin \xi x_1\, e^{i\omega t}, \quad \phi^{(1)} = (V/b)\, e^{i\omega t}. \end{aligned} \tag{81}$$

When independent of x_3, the stress-displacement equations become

$$\begin{aligned} T_p^{(0)} &= 2b[c_{p1}^{(0)}u_{1,1}^{(0)} + c_{p2}^{(0)}u_2^{(1)} + c_{p4}^{(0)}u_3^{(1)} + c_{p5}^{(0)}u_{3,1}^{(0)} + c_{p6}^{(0)}(u_{2,1}^{(0)} + u_1^{(1)}) + e_{1p}^{(0)}\phi_{,1}^{(0)} + e_{2p}^{(0)}\phi^{(1)}], \\ T_p^{(1)} &= (2b^3/3)(c_{p1}^{(1)}u_{1,1}^{(1)} + c_{p5}^{(1)}u_{3,1}^{(1)} + c_{p6}^{(1)}u_{2,1}^{(1)} + e_{1p}^{(1)}\phi_{,1}^{(1)}), \\ D_i^{(0)} &= 2b[e_{i1}^{(0)}u_{1,1}^{(0)} + e_{i2}^{(0)}u_2^{(1)} + e_{i4}^{(0)}u_3^{(1)} + e_{i5}^{(0)}u_{3,1}^{(0)} + e_{i6}^{(0)}(u_{2,1}^{(0)} + u_1^{(1)}) - \epsilon_{i1}\phi_{,1}^{(0)} - \epsilon_{i2}\phi^{(1)}], \\ D_i^{(1)} &= (2b^3/3)(e_{i1}^{(1)}u_{1,1}^{(1)} + e_{i5}^{(1)}u_{3,1}^{(1)} + e_{i6}^{(1)}u_{2,1}^{(1)} - \epsilon_{i1}\phi_{,1}^{(1)}). \end{aligned} \tag{82}$$

When these expressions and those for $T_j^{(0)}$ and $T_j^{(1)}$, from (66), are inserted in the equations of motion (63), the displacement equations of motion become:

$$\begin{aligned} &c_{11}^{(0)}u_{1,11}^{(0)} + c_{12}^{(0)}u_{2,1}^{(1)} + c_{14}^{(0)}u_{3,1}^{(1)} + c_{15}^{(0)}u_{3,11}^{(0)} + c_{16}^{(0)}(u_{2,11}^{(0)} + u_{1,1}^{(1)}) + e_{11}^{(0)}\phi_{,11}^{(0)} + e_{21}^{(0)}\phi_{,1}^{(1)} = (1+R)\rho\ddot{u}_1^{(0)} \\ &c_{61}^{(0)}u_{1,11}^{(0)} + c_{62}^{(0)}u_{2,1}^{(1)} + c_{64}^{(0)}u_{3,1}^{(1)} + c_{65}^{(0)}u_{3,11}^{(0)} + c_{66}^{(0)}(u_{2,11}^{(0)} + u_{1,1}^{(1)}) + e_{16}^{(0)}\phi_{,11}^{(0)} + e_{26}^{(0)}\phi_{,1}^{(1)} = (1+R)\rho\ddot{u}_2^{(0)} \\ &c_{51}^{(0)}u_{1,11}^{(0)} + c_{52}^{(0)}u_{2,1}^{(1)} + c_{54}^{(0)}u_{3,1}^{(1)} + c_{55}^{(0)}u_{3,11}^{(0)} + c_{56}^{(0)}(u_{2,11}^{(0)} + u_{1,1}^{(1)}) + e_{15}^{(0)}\phi_{,11}^{(0)} + e_{25}^{(0)}\phi_{,1}^{(1)} = (1+R)\rho\ddot{u}_3^{(0)} \\ &c_{11}^{(1)}u_{1,11}^{(1)} + c_{15}^{(1)}u_{3,11}^{(1)} + c_{16}^{(1)}u_{2,11}^{(1)} + e_{11}^{(1)}\phi_{,11}^{(1)} \\ &- 3b^{-2}[c_{61}^{(0)}u_{1,1}^{(0)} + c_{62}^{(0)}u_2^{(1)} + c_{64}^{(0)}u_3^{(1)} + c_{65}^{(0)}u_{3,1}^{(0)} + c_{66}^{(0)}(u_{2,1}^{(0)} + u_1^{(1)}) + e_{16}^{(0)}\phi_{,1}^{(0)} + e_{26}^{(0)}\phi^{(1)}] = (1+3R)\rho\ddot{u}_1^{(1)} \\ &c_{61}^{(1)}u_{1,11}^{(1)} + c_{65}^{(1)}u_{3,11}^{(1)} + c_{66}^{(1)}u_{2,11}^{(1)} + e_{16}^{(1)}\phi_{,11}^{(1)} - 3b^{-2}[c_{21}^{(0)}u_{1,1}^{(0)} + c_{22}^{(0)}u_2^{(1)} + c_{24}^{(0)}u_3^{(1)} + c_{25}^{(0)}u_{3,1}^{(0)} + c_{26}^{(0)}(u_{2,1}^{(0)} \\ &+ u_1^{(1)}) + e_{12}^{(0)}\phi_{,1}^{(0)} + e_{22}^{(0)}\phi^{(1)}] = (1+3R)\rho\ddot{u}_2^{(1)} \\ &-3b^{-2}[c_{41}^{(0)}u_{1,1}^{(0)} + c_{42}^{(0)}u_2^{(1)} + c_{44}^{(0)}u_3^{(1)} + c_{45}^{(0)}u_{3,1}^{(0)} + c_{46}^{(0)}(u_{2,1}^{(0)} + u_1^{(1)}) + e_{14}^{(0)}\phi_{,1}^{(0)} + e_{24}^{(0)}\phi^{(1)}] = (1+3R)\rho\ddot{u}_3^{(1)} \\ &e_{11}^{(0)}u_{1,11}^{(0)} + e_{12}^{(0)}u_{2,1}^{(1)} + e_{14}^{(0)}u_{3,1}^{(1)} + e_{15}^{(0)}u_{3,11}^{(0)} + e_{16}^{(0)}(u_{2,11}^{(0)} + u_{1,1}^{(1)}) - \epsilon_{11}\phi_{,11}^{(0)} - \epsilon_{12}\phi_{,1}^{(1)} = D^{(0)} \\ &e_{11}^{(1)}u_{1,11}^{(1)} + e_{15}^{(1)}u_{3,11}^{(1)} + e_{16}^{(1)}u_{2,11}^{(1)} - \varepsilon_{11}\phi_{,1}^{(1)} - 3b^{-2}[e_{21}^{(0)}u_{1,1}^{(0)} + e_{22}^{(0)}u_2^{(1)} + e_{24}^{(0)}u_3^{(1)} \\ &+ e_{25}^{(0)}u_{3,1}^{(0)} + e_{26}^{(0)}(u_{2,1}^{(0)} + u_1^{(1)}) - \epsilon_{21}\phi_{,1}^{(0)} - \epsilon_{22}\phi^{(1)} = D^{(1)}. \end{aligned} \tag{83}$$

We first find a particular solution of (83) for the constant forcing term $\phi^{(1)}$. This is the same as the solution for simple thickness modes, of the preceding section, except that the frequency ω need not be a resonance frequency. Then the amplitudes $A_1{}^0$, $A_2{}^0$, $A_3{}^0$ may be obtained from (70).

The complementary solution is found by substituting (81) into (83) with $\phi^{(1)}$ zero. The eighth of (83) may be set aside as it serves only at a later stage to yield the surface charge and the current through the crystal. In the seventh of (83), $D^{(0)}$ is zero from symmetry; and the equation may be solved for A_7:

$$A_7 = (e^{(0)}_{11}\bar{\xi}A_1 + e^{(0)}_{16}\bar{\xi}A_2 + e^{(0)}_{15}\bar{\xi}A_3 + e^{(0)}_{16}A_4 + e^{(0)}_{12}A_5 + e^{(0)}_{14}A_6)/\epsilon_{11}\bar{\xi} \tag{84}$$

and used to replace A_7 in the remaining six equations. The result is

$$a_{ij}A_j = 0, \quad a_{ij} = a_{ji}; \quad i, j = 1 \ldots 6 \tag{85}$$

where

$$\begin{aligned}
&a_{11} = \tilde{c}_{11}\bar{\xi}^2 - R'\hat{\Omega}^2 && a_{12} = \tilde{c}_{16}\bar{\xi}^2\kappa_6 && a_{23} = \tilde{c}_{56}\bar{\xi}^2\\
&a_{22} = (\tilde{c}_{66}\bar{\xi}^2 - R'\hat{\Omega}/\kappa_6^2)\kappa_6 && a_{13} = \tilde{c}_{15}\bar{\xi}^2 && a_{24} = \tilde{c}_{66}\bar{\xi}\kappa_6\\
&a_{33} = \tilde{c}_{55}\bar{\xi}^2 - R'\hat{\Omega}^2 && a_{14} = \tilde{c}_{16}\bar{\xi}\kappa_6 && a_{25} = \tilde{c}_{26}\bar{\xi}\kappa_2\\
&a_{44} = (\tilde{c}_{66} + \hat{c}_{11}\bar{\xi}^2 - \hat{\Omega}^2/\kappa_6^2)\kappa_6 && a_{15} = \tilde{c}_{12}\bar{\xi}\kappa_2 && a_{26} = \tilde{c}_{46}\bar{\xi}\kappa_4\\
&a_{55} = (\tilde{c}_{22} + \hat{c}_{66}\bar{\xi}^2 - \hat{\Omega}^2/\kappa_2^2)\kappa_2 && a_{16} = \tilde{c}_{14}\bar{\xi}\kappa_4 &&\\
&a_{66} = (\tilde{c}_{44} + \hat{c}_{55}\bar{\xi}^2 - \hat{\Omega}^2/\kappa_4^2)\kappa_4 && &&\\
&a_{34} = \tilde{c}_{56}\bar{\xi}\kappa_6 && a_{35} = \tilde{c}_{25}\bar{\xi}\kappa_2 && a_{36} = \tilde{c}_{45}\bar{\xi}\kappa_4\\
&a_{45} = (\tilde{c}_{26} + \hat{c}_{16}\bar{\xi}^2)\kappa_2 && a_{46} = (\tilde{c}_{46} + \hat{c}_{15}\bar{\xi}^2)\kappa_4 && a_{56} = (\tilde{c}_{24} + \hat{c}_{56}\bar{\xi}^2)\kappa_4
\end{aligned} \tag{86}$$

in which

$$\hat{\Omega}^2 = (\pi^2/12)(1+3R)\omega^2/\omega_0^2, \quad \omega_0^2 = \pi^2 c_{66}/4\rho b^2, \quad R' = 3(1+R)(1+3R),$$
$$\tilde{c}_{pq} = (c_{pq} - e_{1p}e_{1q}/\epsilon_{11})/c_{66}, \quad \hat{c}_{pq} = c^{(1)}_{pq}/3\kappa_r\kappa_s c_{66}, \quad \bar{\xi} = \xi b. \tag{87}$$

In $\hat{c}_{pq}$, when p (or q) = 1, 5, 6 then r (or s) = 2, 4, 6 respectively.

Resonance occurs when the determinant of the coefficients of the A_j, in (85), is equal to zero:

$$|a_{ij}| = 0. \tag{88}$$

This is a sextic equation in $\bar{\xi}^2$ yielding a six-branched dispersion relation as illustrated in Fig. 1 for the SC-out and $R = 0.01$.

In Fig. 1, the ordinate is Ω as given in (30) and the abscissa is $X = 2\xi b/\pi$. The six branches are identified according to the behavior of the corresponding modes at long wave-lengths ($\xi \to 0$):

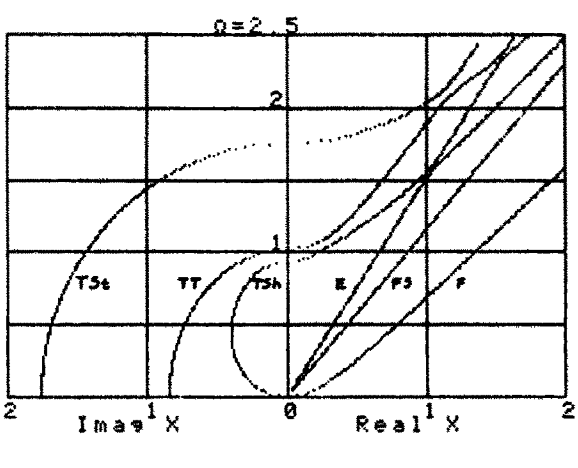

Fig. 1. Plot of dispersion relation (88).

F = flexure
FS = face-shear
E = extension
TSh = thickness-shear
TT = thickness-twist
TSt = thickness-stretch

The computation of the roots of the sextic (88), for Fig. 1, required about 90 hr on the HP-85 microcomputer. In order to reduce the computation time for subsequent use in the interval $0.91 < \Omega < 1.07$, an approximation was introduced. In that interval, each of the six branches was divided into three segments and a cubic, quadratic or linear curve, as appropriate, was fitted to each of the eighteen segments so that the roots along each curve could be expressed explicitly. For each frequency, the time for computation and storage of the six roots was thereby reduced by a factor of about 1250. The result is illustrated in Fig. 2.

The equations of motion are satisfied by (81) and (64) for each of the ξ_n, $n = 1\ldots 6$. Hence, we can write

$$\begin{aligned}
u_1^{(0)} &= \sum_{n=1}^{6} A_{1n} b \sin(\xi_n x_1 + \epsilon)\, e^{i\omega t}, & u_1^{(1)} &= A_1{}^0 + \sum_{n=1}^{6} A_{4n} \cos(\xi_n x_1 + \epsilon)\, e^{i\omega t}\\
u_2^{(0)} &= \sum_{n=1}^{6} A_{2n} b \sin(\xi_n x_1 + \epsilon)\, e^{i\omega t}, & u_2^{(1)} &= A_2{}^0 + \sum_{n=1}^{6} A_{5n} \cos(\xi_n x_1 + \epsilon)\, e^{i\omega t}\\
u_3^{(0)} &= \sum_{n=1}^{6} A_{3n} b \sin(\xi_n x_1 + \epsilon)\, e^{i\omega t}, & u_3^{(1)} &= A_3{}^0 + \sum_{n=1}^{6} A_{6n} \cos(\xi_n x_1 + \epsilon)\, e^{i\omega t}\\
\phi^{(0)} &= \sum_{n=1}^{6} A_{7n} b \sin(\xi_n x_1 + \epsilon)\, e^{i\omega t}, & \phi^{(1)} &= (V/b)\, e^{i\omega t},
\end{aligned} \tag{89}$$

where $\epsilon = 0$ or $\pi/2$.

8. VIBRATIONS OF A STRIP

A strip, bounded by edges at, say, $x_1 = \pm a$, may be subject to a variety of edge conditions. The only one that can be attained physically without difficulty is the traction-free condition. However, the simplest condition mathematically is one for which the six branches of the dispersion relation are not coupled. In the present case, this can be satisfied by

$$u_1^{(0)} = u_2^{(0)} = u_3^{(0)} = \phi^{(0)} = T_1^{(1)} = T_6^{(1)} = T_5^{(1)} = D_1^{(1)} = 0 \quad \text{on} \quad x_1 = \pm a; \tag{90}$$

a combination which is admissible according to (43) and (44). The conditions (90) are the

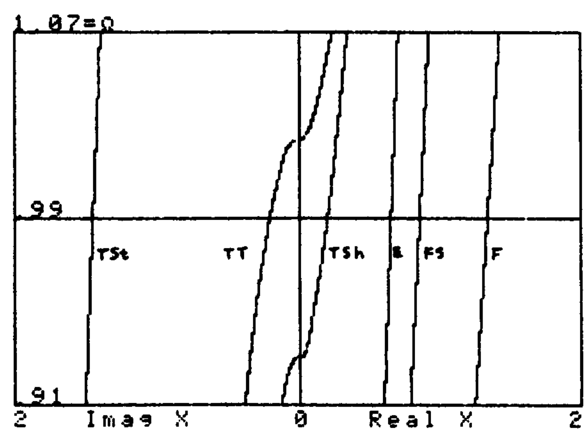

Fig. 2. Portion of dispersion curves of Fig. 1 with expanded frequency scale.

analogue of those for the simply supported beam in the Bernoulli–Euler or Timoshenko beam theory, in which only the bending moment $T_1^{(1)}$ and the deflection $u_2^{(0)}$, of the functions in (90), are present and required to be zero at the ends of the beam.

With (89), (90) can be satisfied by

$$\sin \xi_n a = 0 \tag{91}$$

for $\epsilon = 0$ and

$$\cos \xi_n a = 0 \tag{92}$$

for $\epsilon = \pi/2$ for each branch separately—for real ξ_n. Thus, the boundary conditions (90) can be satisfied by (91) or (92), for real ξ_n and each n separately if

$$X = 2\xi_n b/\pi = mb/a, \quad m = 1, 2, 3\ldots \tag{93}$$

This equation and (88) can be represented graphically by replotting the real part of Fig. 1 or Fig. 2 with the reciprocals of the abscissae of each of the six curves (i.e. a plot of Ω vs a/mb) and then multiplying the resulting abscissae by the integers m to produce a set of equally spaced Ω vs a/b curves (overtone branches) for each of the six replotted dispersion curves. The result is shown in Fig. 3 for the range $0 < a/b < 24$ and the frequency range of Fig. 2. The range $17 < a/b < 23$ is shown in Fig. 4 with the abscissa expanded by a factor of 4. The numbers following *F*, *FS*, *E*, *TSh* and *TT*, in Figs. 3 and 4, are the values of m, in (93), identifying the order of the overtone. No thickness-stretch branches appear, as their ξ is imaginary in the frequency range displayed.

The computation for the strip with free edges is more complicated; but the main effect on the frequency spectrum, of the coupling of the modes and overtones at the free edges, is the elimination of intersections of overtone branches. The general features of the frequency spectrum of the coupled modes can readily be sketched, without computation, over the grid of the spectrum of uncoupled modes and overtones. Figure 5 is such a sketch on the background of Fig. 4. If m odd or even has been chosen correctly for each set of overtones, the only error in a sketch—such as in Fig. 5—is in the strength of the coupling; i.e. whether the coupled branches lie close to the intersections (weak coupling) or far from the intersections (strong coupling).

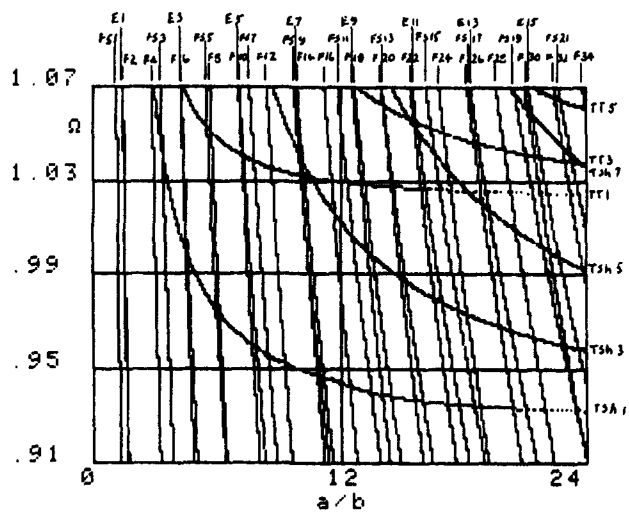

Fig. 3. Frequency spectrum of strip, with mixed edge-conditions (90), computed from (88) and (93).

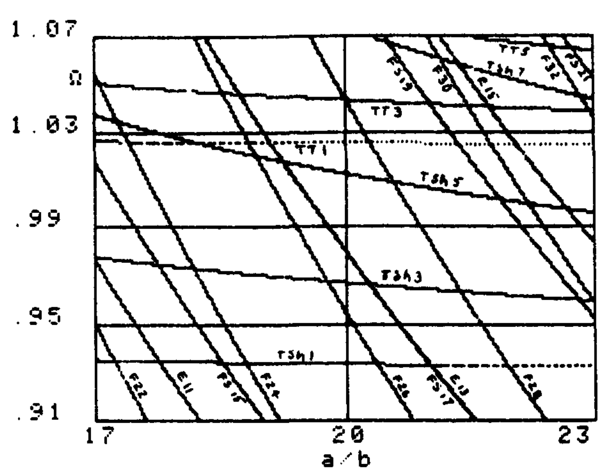

Fig. 4. Portion of frequency spectrum of Fig. 3 with expanded a/b.

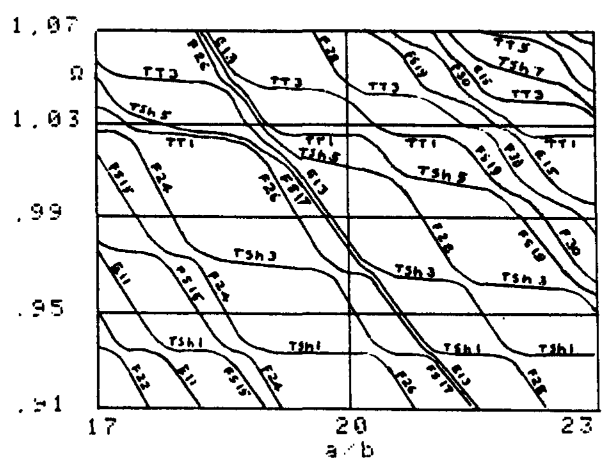

Fig. 5. Frequency spectrum of strip with free edges, sketched, without computation, on grid of branches illustrated in Fig. 4.

In both Figs. 4 and 5, it may be seen that there is an interval of a/b, along the branch TSh-1 between F-24 and F-26, where there are no E and FS overtone branches. Halfway into that interval, at about $a/b = 19.6$, would be a favorable dimensional ratio for a minimum of activity dip resulting from coupling of thickness-shear with flexure, extension and face-shear. Similar ratios occur at $a/b = 32.2, 44.8, 62.1, 71.5\ldots$; again for $R = 0.01$ and the SC-cut.

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APPENDIX

Formulas for material constants of doubly rotated quartz plates

α-Quartz has an axis of three-fold symmetry, say X_3, and three axes of two-fold symmetry, one of which is designated as X_1 in a right-handed, rectangular coordinate system X_i; $i = 1, 2, 3$. Rotate the X_i system a positive angle ϕ about X_3 and a positive angle θ about X_1 to a new orientation x_i. The direction cosines, l_{ij}, of the x_i axes with respect to the X_i axes:

$$\begin{array}{cccc} & X_1 & X_2 & X_3 \\ x_1 & l_{11} & l_{12} & l_{13} \\ x_2 & l_{21} & l_{22} & l_{23} \\ x_3 & l_{31} & l_{32} & l_{33} \end{array} \tag{a}$$

are

$$\begin{array}{lll} l_{11} = \cos\phi & l_{12} = \sin\phi & l_{13} = 0 \\ l_{21} = -\sin\phi\cos\theta & l_{22} = \cos\phi\cos\theta & l_{23} = \sin\theta \\ l_{31} = \sin\phi\sin\theta & l_{32} = -\cos\phi\sin\theta & l_{33} = \cos\theta. \end{array} \tag{b}$$

The quartz plate is cut with its thickness parallel to x_2 and a pair of edges parallel to x_1.

The elastic, c_{rstu}, piezoelectric, e_{rst}, and dielectric, ϵ_{rs}, constants, referred to the rotated axes x_i, expressed in terms of the constants c^0_{ijk}, e^0_{ijk}, ϵ^0_{ij}, referred to the axes X_i, are

$$\begin{aligned} c_{rstu} &= c^0_{ijkl} l_{ri} l_{sj} l_{tk} l_{ul}, \\ e_{rst} &= e^0_{ijk} l_{ri} l_{sj} l_{tk}, \\ \epsilon_{rs} &= \epsilon^0_{ij} l_{ri} l_{sj}, \end{aligned} \tag{c}$$

summed over repeated indices i, j, k, l. For α-quartz,

$$\begin{aligned} & c^0_{22} = c^0_{11}, \quad c^0_{23} = c^0_{13}, \quad c^0_{24} = -c^0_{14} = -c^0_{56}, \quad c^0_{55} = c^0_{44}, \quad c^0_{66} = (c^0_{11} - c^0_{12})/2. \\ & c^0_{15} = c^0_{16} = c^0_{25} = c^0_{26} = c^0_{34} = c^0_{35} = c^0_{36} = c^0_{45} = c^0_{46} = 0, \\ & e^0_{11} = -e^0_{12} = -e^0_{26}, \quad e^0_{14} = -e^0_{25}, \\ & e^0_{13} = e^0_{15} = e^0_{16} = e^0_{21} = e^0_{22} = e^0_{23} = e^0_{24} = e^0_{31} = e^0_{32} = e^0_{33} = e^0_{34} = e^0_{35} = e^0_{36} = 0, \\ & \epsilon^0_{11} = \epsilon^0_{22}, \quad \epsilon^0_{12} = \epsilon^0_{23} = \epsilon^0_{31} = 0. \end{aligned} \tag{d}$$

Thus, we have, from (c) and (d),

$$\begin{aligned} c_{rstu} = {} & c^0_{11}\left[l_{r1}l_{s1}l_{t1}l_{u1} + l_{r2}l_{s2}l_{t2}l_{u2} + \frac{1}{2}(l_{r1}l_{s2} + l_{r2}l_{s1})(l_{t1}l_{u2} + l_{t2}l_{u1}) \right] \\ & + c^0_{12}\left[l_{r1}l_{s1}l_{t2}l_{u2} + l_{r2}l_{s2}l_{t1}l_{u1} - \frac{1}{2}(l_{r1}l_{s2} + l_{r2}l_{s1})(l_{t1}l_{u2} + l_{t2}l_{u1}) \right] \\ & + c^0_{13}(l_{r1}l_{s1}l_{t3}l_{u3} + l_{r3}l_{s3}l_{t1}l_{u1} + l_{r2}l_{s2}l_{t3}l_{u3} + l_{r3}l_{s3}l_{t2}l_{u2}) \\ & + c^0_{14}[(l_{r1}l_{s1} - l_{r2}l_{s2})(l_{t2}l_{u3} + l_{t3}l_{u2}) + (l_{t1}l_{u1} - l_{t2}l_{u2})(l_{r2}l_{s3} + l_{r3}l_{s2}) \\ & \qquad + (l_{r3}l_{s1} + l_{r1}l_{s3})(l_{t1}l_{u2} + l_{t2}l_{u1}) + (l_{r1}l_{s2} + l_{r2}l_{s1})(l_{t3}l_{u1} + l_{t1}l_{u3})] \\ & + c^0_{33}(l_r l_{s3} + l_{t3}l_{u3}) \\ & + c^0_{44}[(l_{r2}l_{s3} + l_{r3}l_{s2})(l_{t2}l_{u3} + l_{t3}l_{u2}) + (l_{r3}l_{s1} + l_{r1}l_{s3})(l_{t3}l_{u1} + l_{t1}l_{u3})], \end{aligned}$$

$$\begin{aligned} e_{rst} = {} & e^0_{11}(l_{r1}l_{s1}l_{t1} - l_{r1}l_{s2}l_{t2} - l_{r2}l_{s1}l_{t2} - l_{r2}l_{s2}l_{t1}) \\ & + e^0_{14}(l_{r1}l_{s2}l_{t3} + l_{r1}l_{s3}l_{t2} - l_{r2}l_{s3}l_{t1} - l_{r2}l_{s1}l_{t3}), \\ \epsilon_{rs} = {} & \epsilon^0_{11}(l_{r1}l_{s1} + l_{r2}l_{s2}) + \epsilon^0_{33}l_{43}l_{s3}. \end{aligned}$$

Bechmann's values for α-quartz are[8]

$$\begin{array}{llll} c^0_{11} = 86.74 & c^0_{14} = -17.91 & e^0_{11} = 0.171 & e^0_{14} = -0.0406 \\ c^0_{12} = 6.98 & c^0_{33} = 107.2 & \epsilon^0_{11} = 39.21 & \epsilon^0_{33} = 41.03 \\ c^0_{13} = 11.91 & c^0_{44} = 57.94 & & \end{array}$$

in units of $10^9 N/m^2$, C/m^2 and $10^{-12} F/m$ for the c^0_{pq}, e^0_{ip} and ϵ^0_{ij}, respectively.

For the SC-cut quartz plate ($\phi = 21.93°$, $\theta = 33.93°$), the constants are listed in Table 1.

Table 1. Doubly-rotated quartz plate

```
Φ= 21.93°   θ= 33 93°

     CONSTANT                UNITS
Cpq=ELASTIC STIFFNESS· 10^9 N/m²
Eip=PIEZOELECTRIC CONSTANT: C/m²
Kij=DIELECTRIC CONST:10^-12 F/m

C11= 86.7400000001
C12= 1.71298689814
C13= 17.177013102
C14=-.484711423
C15=-13.5533039091
C16=-9.1177467817
C22= 115.702789532
C23=-3.8751539075
C24= 8.8720683789
C25= .8851190534
C26= 18.8309418361
C33= 109.807518283
C34= 3.3715867534
C35= 12.6681848557
C36=-9.71319505441
C44= 42.1548460928
C45=-9.71319505431
C46= .88511905334
C55= 59.1161712855
C56= 5.59622323258
C66= 38.7038287147

E11= 7.01240600627E-2
E12=-8.58822822078E-2
E13= 1.57582221448E-2
E14= .017175794557
E15= 8.70538637456E-2
E16=-.129403404157
E21=-.129403404157
E22= 8.90858979159E-2
E23= 4.03175062407E-2
E24=-5.99309706679E-2
E25= 6.04272390907E-2
E26=-2.94726979707E-2
E31= 8.70538637458E-2
E32=-5.99309706681E-2
E33=-2.71228930779E-2
E34= 4.03175062408E-2
E35=-4.06513620922E-2
E36= 1.98272390908E-2

K11= 39.21
K22= 39.7770473805
K23= .8429018334
K33= 40.4629526195
K12=K13=0
```